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An auto-Bäcklund transformation and exact solutions for Wick-type stochastic generalized KdV equations*

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Abstract
Wick-type stochastic generalized KdV equations are researched. By using the homogeneous balance, an auto-Bäcklund transformation to the Wick-type stochastic generalized KdV equations is derived. And stochastic single soliton and stochastic multi-soliton solutions are shown by using the Hermite transform.

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1. Introduction

In this paper we will give exact solutions of Wick-type stochastic generalized KdV equations in the following form:

$$U_t + G_1(t) \diamond [6U \diamond U_x + U_{xxx}] + 6F_1(t) \diamond U = x[F_1'(t) + 12F_1(t)^{\diamond 2} \diamond G_1(t)] \tag{1.1}$$

where $F_1(t) = [f(t) + K_1 B(t)]$ and $G_1(t) = g(t) + K_2 W(t)$, $f(t)$ and $g(t)$ are square integrable or bounded functions of t, $B(t)$ is a Brownian motion, $W(t)$ is Gaussian white noise, i.e. $W(t) = \dot{B}(t)$, K_1 and K_2 are some constants. In fact, we hope to give exact solutions of generalized KdV equations with random coefficients $B(t)$ and/or $W(t)$, in this case $f(t) = g(t) = 0$.

(1.1) is the perturbation of the coefficients $f(t)$ and $g(t)$ of the generalized KdV equation

$$u_t + g(t)[6uu_x + u_{xxx}] + 6f(t)u = x[f'(t) + 12g(t)f^2(t)] \tag{1.2}$$

by $K_1 B(t)$ and $K_2 W(t)$, respectively. (1.2) was discussed by M L Wang *et al* in [13]. They gave the exact solutions of (1.2) by using the homogeneous balance principle which was given by M L Wang in [11]. The homogeneous balance method has been widely applied

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to derive the nonlinear transformations and exact solutions (especially the solitary waves), and auto-Bäcklund transformations as well as the similarity reductions of nonlinear PDEs in mathematical physics. These subjects have been researched by many authors, such as M L Wang [11], M L Wang and Y M Wang [12], M L Wang *et al* [13], E G Fan [6, 7], etc.

Random waves are an important subject of stochastic partial differential equations. Now, stochastic KdV equations have been studied by many authors. As far as we know they are A de Bouard and A Debussche [2, 3], A Debussche and J Printems [4, 5], V V Konotop and L Vźquez [9], J Printems [10], Y C Xie [14, 15] and so on. In [8], H Holden *et al* gave the white noise functional approach to research stochastic partial differential equations in Wick versions. As Xie did in [14] and [15], we will use the white noise method to give auto-Bäcklund transformations and exact solutions of Wick-type stochastic generalized KdV equation (1.1).

2. SPDEs driven by white noise

In this section we will summarize the main matters for stochastic partial differential equations which use the white noise functional approach. Please see H Holden *et al*'s book [8] for details.

Let $h_n(x)$ be the Hermite polynomials. Put $\xi_n(x) = e^{-\frac{1}{2}x^2} h_n(\sqrt{2}x)/(\pi(n-1)!)^{1/2}$, $n \geqslant 1$. We have that the collection $\{\xi_n\}_{n\geqslant 1}$ constitutes an orthogonal basis for $L^2(\mathbb{R})$ and $\sup_{x\in\mathbb{R}} |\xi_n(x)| = O\left(\frac{1}{n^{1/12}}\right)$.

If we denote $\alpha = (\alpha_1, \ldots, \alpha_d)$ being d-dimensional multi-indices with $\alpha_1, \ldots, \alpha_d \in \mathbb{N}$, we have that the family of tensor products $\xi_\alpha = \xi_{(\alpha_1,\ldots,\alpha_d)} = \xi_{\alpha_1} \otimes \cdots \otimes \xi_{\alpha_d} (\alpha \in \mathbb{N}^d)$ forms an orthogonal basis for $L^2(\mathbb{R}^d)$. Let $\alpha^{(i)} = \left(\alpha_1^{(i)}, \ldots, \alpha_d^{(i)}\right)$ be the ith multi-index number in some fixed ordering of all d-dimensional multi-indices $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$. We can, and will, assume that this ordering has the property that

$$i < j \Rightarrow \alpha_1^{(i)} + \cdots + \alpha_d^{(i)} \leqslant \alpha_1^{(j)} + \cdots + \alpha_d^{(j)}.$$

Now define

$$\eta_i = \xi_{\alpha^{(i)}} = \xi_{\alpha_1^{(i)}} \otimes \cdots \otimes \xi_{\alpha_d^{(i)}} \qquad i \geqslant 1.$$

We denote multi-indices as elements of the space $\left(\mathbb{N}_0^{\mathbb{N}}\right)_c$ of all sequences $\alpha = (\alpha_1, \alpha_2, \ldots)$ with elements $\alpha_i \in \mathbb{N}_0$ and with compact support, i.e. with only finitely many $\alpha_i \neq 0$. We denote $\mathcal{J} = \left(\mathbb{N}_0^{\mathbb{N}}\right)_c$. For $\alpha = (\alpha_1, \alpha_2, \ldots,) \in \mathcal{J}$, we define

$$H_\alpha(\omega) = \prod_{i=1}^{\infty} h_{\alpha_i}(\langle \omega, \eta_i \rangle) \qquad \omega \in (S(\mathbb{R}^d))^*.$$

With $n \in \mathbb{N}$, let $(S)_1^n$ consist of those $x = \sum_\alpha c_\alpha H_\alpha \in \bigoplus_{k=1}^n L^2(\mu)$ with $c_\alpha \in \mathbb{R}^n$ such that $\|x\|_{1,k}^2 = \sum_\alpha c_\alpha^2 (\alpha!)^2 (2\mathbb{N})^{k\alpha} < \infty, \forall k \in \mathbb{N}$ with $c_\alpha^2 = |c_\alpha|^2 = \sum_{k=1}^n \left(c_\alpha^{(k)}\right)^2$ if $c_\alpha = \left(c_\alpha^{(1)}, \ldots, c_\alpha^{(n)}\right) \in \mathbb{R}^n$. where μ is the white noise measure on $(S^*(\mathbb{R}), \mathcal{B}(S^*(\mathbb{R})))$, $\alpha! = \prod_{k=1}^\infty \alpha_k!$ and $(2\mathbb{N})^\alpha = \prod_j (2j)^{\alpha_j}$ for $\alpha = (\alpha_1, \alpha_2, \ldots,) \in \mathcal{J}$, where $(S(\mathbb{R}^d))$ and $(S(\mathbb{R}^d))^*$ are the Hida test function space and the Hida distribution space on $\mathbb{R}^d$, respectively.

The space $(S)_{-1}^n$ consists of all formal expansions $X = \sum_\alpha b_\alpha H_\alpha$ with $b_\alpha \in \mathbb{R}^n$ such that $\|X\|_{-1,-q} = \sum_\alpha b_\alpha^2 (2\mathbb{N})^{-q\alpha} < \infty$ for some $q \in \mathbb{N}$. The family of seminorms $\|x\|_{1,k}, k \in \mathbb{N}$ gives rise to a topology on $(S)_1^n$, and we can regard $(S)_{-1}^n$ as the dual of $(S)_1^n$ by the action

$$\langle X, x \rangle = \sum_\alpha (b_\alpha, c_\alpha)\alpha!$$

and (b_α, c_α) is the usual inner product in $\mathbb{R}^n$.

For $X = \sum_\alpha a_\alpha H_\alpha$, $Y = \sum_\alpha b_\alpha H_\alpha \in (S)^n_{-1}$ with $a_\alpha, b_\alpha \in \mathbb{R}^n$

$$X \diamond Y = \sum_{\alpha,\beta} (a_\alpha, b_\beta) H_{\alpha+\beta}$$

is called the Wick product of X and Y.

We can prove that the spaces $(S(\mathbb{R}^d))$, $(S(\mathbb{R}^d))^*$, $(S)_1$ and $(S)_{-1}$ are closed under Wick products.

For $X = \sum_\alpha a_\alpha H_\alpha \in (S)^n_{-1}$ with $a_\alpha \in \mathbb{R}^n$, the Hermite transform of X, denoted by $\mathcal{H}(X)$ or $\widetilde{X}$, is defined by

$$\mathcal{H}(X) = \widetilde{X}(z) = \sum_\alpha a_\alpha z^\alpha \in \mathbb{C}^n \quad \text{(when convergent)}$$

where $z = (z_1, z_2, \cdots) \in \mathbb{C}^{\mathbb{N}}$ (the set of all sequences of complex numbers) and $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} \cdots$ for $\alpha = (\alpha_1, \alpha_2, \cdots) \in \mathcal{J}$.

For $X, Y \in (S)^N_{-1}$, by this definition we have

$$\widetilde{X \diamond Y}(z) = \widetilde{X}(z) \cdot \widetilde{Y}(z)$$

for all z such that $\widetilde{X}(z)$ and $\widetilde{Y}(z)$ exist. The product on the right-hand side of the above formula is the complex bilinear product of two elements of $\mathbb{C}^{\mathbb{N}}$ defined by $\left(z_1^1, \ldots, z_n^1\right) \cdot \left(z_1^2, \ldots, z_n^2\right) = \sum_{k=1}^n z_k^1 z_k^2$, where $z_k^i \in \mathbb{C}$.

Let $X = \sum_\alpha a_\alpha H_\alpha \in (S)^n_{-1}$. Then the vector $c_0 = \widetilde{X}(0) \in \mathbb{R}^n$ is called the generalized expectation of X and is denoted by $E(X)$. Suppose that $f : V \to \mathbb{C}^m$ is an analytic function, where V is a neighbourhood of $E(X)$. Assume that the Taylor series of f around $E(X)$ has coefficients in $\mathbb{R}^n$. Then the Wick version $f^\diamond(X) = \mathcal{H}^{-1}(f \circ \widetilde{X}) \in (S)^m_{-1}$.

The Wick exponential of $X \in (S)_{-1}$ is defined by $\exp^\diamond\{X\} = \sum_{n=0}^\infty X^{\diamond n}/n!$. Using the Hermite transform we have that the Wick exponential has the same algebraic properties as the usual exponential. For example, $\exp^\diamond\{X + Y\} = \exp^\diamond\{X\} \diamond \exp^\diamond\{Y\}$.

Suppose that modelling considerations lead us to consider an SPDE expressed formally as $A(t, x, \partial_t, \nabla_x, U, \omega) = 0$, where A is some given function, $U = U(t, x, \omega)$ is the unknown (generalized) stochastic process and where the operators $\partial_t = \frac{\partial}{\partial t}$, $\nabla_x = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\right)$ when $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. First we interpret all products as Wick products and all functions as their Wick versions. We indicate this as

$$A^\diamond(t, x, \partial_t, \nabla_x, U, \omega) = 0. \tag{2.1}$$

Secondly, we take the Hermite transform of (2.1). This turns Wick products into ordinary products (of complex numbers) and the equation takes the form

$$\widetilde{A}(t, x, \partial_t, \nabla_x, \widetilde{U}, z_1, z_2, \cdots) = 0 \tag{2.2}$$

where $\widetilde{U} = \mathcal{H}(U)$ is the Hermite transform of U and $z_1, z_2, \ldots,$ are complex numbers. Suppose we can find a solution $u = u(t, x, z)$ of the equation $\widetilde{A}(t, x, \partial_t, \nabla_x, u, z) = 0$ for each $z = (z_1, z_2, \ldots,) \in \mathbb{K}_q(r)$ for some q, r, where $\mathbb{K}_q(r) = \{z = (z_1, z_2, \ldots,) \in \mathbb{C}^{\mathbb{N}}$ and $\sum_{\alpha \neq 0} |z^\alpha|^2 (2\mathbb{N})^{q\alpha} < r^2\}$. Then, under certain conditions, we can take the inverse Hermite transform $U = \mathcal{H}^{-1}u \in (S)_{-1}$ and thereby obtain a solution U of the original Wick equation (2.1). We have the following theorem, which was proved by Holden *et al* [8].

Theorem 2.1. *Suppose $u(t, x, z)$ is a solution (in the usual strong, pointwise sense) of the equation (2.2) for (t, x) in some bounded open set $\mathbf{G} \subset \mathbb{R} \times \mathbb{R}^d$, and for all $z \in \mathbb{K}_q(r)$, for some q, r. Moreover, suppose that $u(t, x, z)$ and all its partial derivatives, which are involved in (2.2), are bounded for $(t, x, z) \in \mathbf{G} \times \mathbb{K}_q(r)$, continuous with respect to $(t, x) \in \mathbf{G}$ for all $z \in \mathbb{K}_q(r)$ and analytic with respect to $z \in \mathbb{K}_q(r)$, for all $(t, x) \in \mathbf{G}$.*

Then there exists $U(t,x) \in (S)_{-1}$ such that $u(t,x,z) = (\widetilde{U}(t,x))(z)$ for all $(t,x,z) \in \mathbf{G} \times \mathbb{K}_q(r)$ and $U(t,x)$ solves (in the strong sense in $(S)_{-1}$) the equation (2.1) in $(S)_{-1}$.

3. Single soliton solutions of stochastic KdV equations

In this and the next section, we will use theorem 2.1 with $d = 1$ to give exact solutions of (1.1).

Taking the Hermite transform of (1.1), we can get the equation

$$\begin{aligned}\widetilde{U}_t(t,x,z) + [g(t) + K_2\widetilde{W}(t,z)][6\widetilde{U}(t,x,z)\widetilde{U}_x(t,x,z) + \widetilde{U}_{xxx}(t,x,z)]\\ + 6[f(t) + K_1\widetilde{B}(t,x)]\widetilde{U}(t,x,z)\\ = x\{[f'(t) + K_1\widetilde{W}(t,z)] + 12[g(t) + K_2\widetilde{W}(t,z)][f(t) + K_1\widetilde{B}(t,z)]^2\}\end{aligned} \tag{3.1}$$

where the Hermite transform of $W(t)$ and $B(t)$ are defined by $\widetilde{W}(t,z) = \sum_{k=1}^{\infty}\int_0^x \eta_k(s)ds z_k$ and $\widetilde{B}(t,z) = \sum_{k=1}^{\infty}\eta_k(t)z_k$, respectively, when $z = (z_1, z_2, \ldots,) \in (\mathbb{C}^{\mathbb{N}})_c$ is a parameter. We first solve the equation (3.1).

For simplicity, denote $F(t,z) = f(t) + K_1\widetilde{B}(t,x)$, $G(t,z) = g(t) + K_2\widetilde{W}(t,z)$ and $u(t,\ x,\ z) = \widetilde{U}(t,\ x,\ z)$. Suppose that the solution of (3.1) is the form

$$u(t,x,z) = \frac{\partial^2 K(\varphi(t,x,z))}{\partial x^2} + V(t,x,z) = K''\varphi_x^2 + K'\varphi_{xx} + V(t,x,z)$$

where $K = K(\varphi)$ is a function of one variable only, $V(t,x,z)$ is a given solution of (3.1) for any $z \in (\mathbb{C}^{\mathbb{N}})_c$, which may be a trivial one, a constant one, and so on. For any $z \in (\mathbb{C}^{\mathbb{N}})_c$, according to the homogeneous balance principle (see [12]), using what Wang *et al* did in [13] we can get the Bäcklund transformation of (3.1) as the follows:

$$u(t,x,z) = 2(\log(\varphi))_{xx} + V(t,x,z) \tag{3.2}$$

$$\begin{aligned}\varphi[\varphi_{tx} + G(\varphi_{xxxx} + 6V\varphi_{xx} + 6F\varphi_x)] - \varphi_x[\varphi_t + G(\varphi_{xxx} + 6V\varphi_x)]\\ + 3G\left(\varphi_{xx}^2 - \varphi_x\varphi_{xxx}\right) = 0.\end{aligned} \tag{3.3}$$

For any fixed $z \in (\mathbb{C}^{\mathbb{N}})_c$, using the Bäcklund transformation (3.2) and (3.3) we can obtain the stochastic solitary wave solutions of (3.1). Since (3.3) is nonlinear, it is difficult to solve it in general, especially when $V(t,x,z)$ is a general function. However, taking $V(t,x,z) = xF(t,z)$ for a solution of (3.3), we can solve the equation

$$\varphi[\varphi_t + G(\varphi_{xxx} + 6xF\varphi_x)]_x - \varphi_x[\varphi_t + G(\varphi_{xxx} + 6xF\varphi_x)] + 3G\left(\varphi_{xx}^2 - \varphi_x\varphi_{xxx}\right) = 0 \tag{3.4}$$

and we get the solution which is an exponential function

$$\varphi(t,x,z) = 1 + \exp\{\phi(t,x,z)\} \tag{3.5}$$

where

$$\phi(t,x,z) = \gamma x A(t,z) - \gamma^3\int_0^t G(s,z)A^3(s,z)\,\mathrm{d}s + x_0 \tag{3.6}$$

and

$$A(t,z) = \exp\left\{-6\int_0^t F(s,z)G(s,z)\,\mathrm{d}s\right\} \tag{3.7}$$

where γ and x_0 are arbitrary constants. Substituting (3.5) with (3.6), (3.7) and $V(t,x,z) = xF(t,z)$ into (3.2) yields that a single solitary wave solution of (3.1) is

$$u(t,x,z) = \frac{\gamma^2 A^2(t,z)\exp\{\phi(t,x,z)\}}{[1 + \exp\{\phi(t,x,z)\}]^2} + xF(t,z). \tag{3.8}$$

By (3.6)–(3.8), $\widetilde{B}(t,z) = \sum_{n=1}^{\infty} z_n \int_0^t \eta_n(s)ds$ and $\widetilde{W}(t,z) = \sum_{n=1}^{\infty} \eta_n(t) z_n$ for $z = (z_1, z_2, \ldots,) \in (\mathbb{C}^{\mathbb{N}})_c$, if we choose the bounded open set $\mathbf{G} \subset \mathbb{R}_+ \times \mathbb{R}, q > 0$ and $|z_j| < (2j)^{-q}$ for all $j \geqslant 1$, then there exists $r > 0$ such that $u(t,x,z), u_t(t,x,z), u_x(t,x,z)$ and $u_{xxx}(t,x,z)$ are uniformly bounded for $(t,x,z) \in \mathbf{G} \times \mathbb{K}_q(r)$, continuous with respect to $(t,x) \in \mathbf{G}$ for all $z \in \mathbb{K}_q(r)$ and analytic with respect to $z \in \mathbb{K}_q(r)$ for all $(t,x) \in \mathbf{G}$. Theorem 2.1 implies that there exists $U(t,x) \in (S)_{-1}$ such that $u(t,x,z) = (\mathcal{H}U(t,x))(z)$ for all $(t,x,z) \in \mathbf{G} \times \mathbb{K}_q(r)$ and that $U(t,x)$ solves the equation (1.1). From the above, we have that $U(t,x)$ is the inverse Hermite transformation of $u(t,x,z)$. Hence, by (3.6), (3.7) and (3.8), we have that a stochastic single solitary solution of (1.1) is

$$U(t,x) = \frac{\gamma^2 A^{\diamond 2}(t,x) \diamond \exp^{\diamond}\{\Phi(t,x)\}}{(1+\exp^{\diamond}\{\Phi(t,x)\})^{\diamond 2}} + x[f(t) + K_1 B(t)] \tag{3.9}$$

where

$$\Phi(t,x) = \gamma x A(t) - \gamma^3 \int_0^t G(s) \diamond A^{\diamond 3}(s)ds + x_0 \tag{3.10}$$

and

$$A(t) = \exp\left\{-6\int_0^t F(s) \diamond G(s)ds\right\}. \tag{3.11}$$

The following special cases are interesting:

(i) Taking $f(t) = 1, g(t) = K_1 = 0$ and $K_2 \neq 0$, we have $\int_0^t F(s) \diamond G(s)ds = K_2 B(t)$, $A(t) = \exp\{-6K_2 B(t)\}$. $\exp^{\diamond}\{B(t)\} = \exp\left\{B(t) - \frac{1}{2}t^2\right\}$ (see lemma 2.6.16 in [8]) implies $A^{\diamond}(t) = \exp^{\diamond}\{-6K_2 B(t)\} = \exp\{-K_2(6B(t) - 3t^2)\}$ and

$$\begin{aligned}\Phi(t,x) &= \gamma x \exp\{-6K_2 B(t)\} - \gamma^3 K_2 \int_0^t \exp\{-9K_2(2B(s) - s^2)\} \diamond W_s ds + x_0 \\ &= \gamma x \exp\{-6K_2 B(t)\} - \gamma^3 K_2 \int_0^t \exp\{-9K_2(2B(s) - s^2)\}\delta B(s) + x_0.\end{aligned}$$

Hence, the stochastic generalized KdV equation with the coefficient $W(t)$

$$U_t + K_2 W(t) \diamond [6U \diamond U_x + U_{xxx}] + 6U = 12K_2 x W(t) \tag{3.12}$$

has the solution

$$\begin{aligned}&U_1(t,x) \\ &= \frac{\gamma^2 \exp^{\diamond}\left\{-12K_2B(t) + \gamma x\exp\{-6K_2B(t)\} - \gamma^3K_2\int_0^t \exp\{-9K_2(2B(s)-s^2)\}\delta B(s) + x_0\right\}}{\left(1+\exp^{\diamond}\left\{\gamma x\exp\{-6K_2B(t)\} - \gamma^3K_2\int_0^t \exp\{-9K_2(2B(s)-s^2)\}\delta B(s) + x_0\right\}\right)^{\diamond 2}} + x.\end{aligned}$$

Where the stochastic integral $\int_0^t \exp\{-9K_2(2B(s) - s^2)\}\delta B(s)$ is a Skorohod integral.

(ii) Choosing $f(t) = g(t) = 0$ and $k_{12} \hat{=} K_1 K_2 \neq 0$, we have

$$\begin{aligned}&\int_0^t F(s) \diamond G(s)ds = k_{12}\int_0^t B_s \diamond W_s ds = \frac{k_{12}}{2}(B^2(t) - t) \\ &A(t) = \exp\{-3k_{12}(B^2(t) - t)\}\end{aligned}$$

and

$$\Phi(t,x) = \gamma x \exp\{-3k_{12}(B^2(t) - t)\} - \gamma^3 k_{12} \int_0^t \exp^{\diamond}\{-9k_{12}(B^2(s) - s)\}\delta B(s) + x_0.$$

Hence, the stochastic generalized KdV equation with the coefficients $B(t)$ and $W(t)$

$$\begin{aligned}&U_t + K_2 W(t) \diamond [6U \diamond U_x + U_{xxx}] + 6K_1 B(t) \diamond U \\ &\quad = x\left[K_2 W(t) + 12K_1 K_2^2 W(t) \diamond B^2(t)\right]\end{aligned} \tag{3.13}$$

has the solution

$$U_1(t,x) = \frac{\gamma^2 e^{\diamond\{-6k_{12}(B^2(t)-t)+\gamma x \exp\{-3k_{12}(B^2(t)-t)\}-\gamma^3 k_{12}\int_0^t \exp^\diamond\{-9k_{12}(B^2(s)-s)\}\delta B(s)+x_0\}}}{\left(1+e^{\diamond\{\gamma x \exp\{-3k_{12}(B^2(t)-t)\}-\gamma^3 k_{12}\int_0^t \exp^\diamond\{-9k_{12}(B^2(s)-s)\}\delta B(s)+x_0\}}\right)^{\diamond 2}} + B(t)x.$$

(iii) Taking $f(t) = g(t) = K_1 = 0$ and $K_2 \neq 0$, we have $\int_0^t F(s)\diamond G(s) = 0$, $A(t,x) = 1$ and

$$\Phi(t,x) = \gamma x - K_2\gamma^3 B(t) + x_0.$$

The stochastic generalized KdV equation with the coefficient $W(t)$

$$U_t + K_2 W(t)\diamond[6U\diamond U_x + U_{xxx}] = 0 \tag{3.14}$$

has the solution

$$\begin{aligned} U_3(t,x) &= \frac{\gamma^2 \exp^\diamond\{\gamma x - \gamma^3 K_2 B(t) + x_0\}}{(1+\exp^\diamond\{\gamma x - \gamma^3 K_2 B(t) + x_0\})^{\diamond 2}} \\ &= \frac{\gamma^2 \exp\left\{\gamma x - \gamma^3 K_2\left(B(t) - \frac{1}{2}t^2\right) + x_0\right\}}{\left(1+\exp\left\{\gamma x - \gamma^3 K_2\left(B(t)-\frac{1}{2}t^2\right)+x_0\right\}\right)^{\diamond 2}} \\ &= \frac{\gamma^2 \exp\left\{\gamma x - \gamma^3 K_2\left(B(t) - \frac{1}{2}t^2\right) + x_0\right\}}{\left(1+\exp\left\{\gamma x - \gamma^3 K_2\left(B(t)-t^2\right)+x_0\right\}\right)^{2}}. \end{aligned}$$

4. Multi-soliton solutions of stochastic generalized KdV equations

In order to obtain general multi-soliton solutions of (1.1), as M L Wang and Y M Wang did in [13], we use the ε-expansion method to solve the equation (3.4), that is, suppose that

$$\varphi(t,x,z) = 1 + \varphi^{(1)}\varepsilon + \varphi^{(2)}\varepsilon^2 + \varphi^{(3)}\varepsilon^3 + \cdots \tag{4.1}$$

where $\varphi^{(k)}$ $(k \geqslant 1)$ to be undetermined, ε is a parameter (for simplicity, we may take $\varepsilon = 1$). Substituting (4.1) into (3.4), collecting all terms with the same order in ε together and setting each coefficient of ε^k $(k \geqslant 1)$ to zero, yields a hierarchy of equations for $\varphi^{(k)}$ $(k \geqslant 1)$, and solving this hierarchy of equations for $\varphi^{(k)}$ to get an exact solution of (3.4) in the form:

$$\varphi(t,x,z) = 1 + \sum_{k=1}^{N}\phi_k + \sum_{i\neq j} a_{ij}\phi_i\phi_j + \sum_{i\neq j\neq k} a_{ijl}\phi_i\phi_j\phi_k + \cdots + a_{12\cdots N}\prod_{k=1}^{N}\phi_k \tag{4.2}$$

where

$$\phi_k(t,x,z) = \exp\left\{\gamma_k A(t,x,z)x - \gamma_k^3\int_0^t G(s,x,z)A^3(s,x,z)\,ds + x_k\right\} \tag{4.3}$$

where $A(t,x,z)$ is expressed by (3.7), $\gamma_k, x_k, a_k, a_{ij}, \ldots, a_{12\cdots N}$ are arbitrary constants, N is a positive integer. Substituting (4.2) with (4.3) and $V(t,x,z) = xF(t,z)$ into (3.2) yields the stochastic N-soliton solution of (3.1)

$$u(t,x,z) = 2(\log(\varphi(t,x,z)))_{xx} + xF(t,z) \tag{4.4}$$

which represents the interaction of stochastic N-solitary waves for any $z \in (\mathbb{C}^{\mathbb{N}})_c$.

As in the stochastic one-soliton solution case, using theorem 2.1, (4.2), (4.3) and (4.4) we have that the stochastic N-soliton solution of (1.1) is

$$U(t,x) = 2(\log^\diamond(\Psi(t,x)))_{xx} + xF(t) \tag{4.5}$$

where

$$\Psi(t,x) = 1 + \sum_{k=1}^{N}\Phi_k + \sum_{i\neq j} a_{ij}\Phi_i \diamond \Phi_j + \sum_{i\neq j\neq k} a_{ijk}\Phi_i \diamond \Phi_j \diamond \Phi_k + \cdots + a_{12\cdots N}\prod_{k=1}^{\diamond N}\Phi_k \quad (4.6)$$

and

$$\Phi_k(t,x) = \exp^{\diamond}\left\{\gamma_k A(t,x)x - \gamma_k^3\int_0^t G(s,x)\diamond A^{\diamond 3}(s,x)ds + x_k\right\}. \quad (4.7)$$

As an illustrative example we will give detailed discussion in the case $N = 2$. Taking

$$\varphi^{(1)}(t,x,z) = \phi_1(t,x,z) + \phi_2(t,x,z)$$

as the discussion of M L Wang *et al* in [13] we have

$$\varphi^{(2)}(t,x,z) = \frac{(\gamma_1-\gamma_2)^2}{(\gamma_1+\gamma_2)^2}\phi_1(t,x,z)\phi_2(t,x,z)$$

and $\varphi^{(k)}(t,x,z) = 0$ for $k = 3,\ldots,N$. If we choose $\varepsilon = 1$, (4.2) yields that the solution of (3.4) is the following:

$$\varphi(t,x,z) = 1 + \phi_1(t,x,z) + \phi_2(t,x,z) + \frac{(\gamma_1-\gamma_2)^2}{(\gamma_1+\gamma_2)^2}\phi_1(t,x,z)\phi_2(t,x,z). \quad (4.8)$$

Substituting (4.8) and $V(t,x,z) = xF(t,z)$ into (3.2) we get the exact solution of (3.1)

$$u(t,x,z) = 2A^2(t,x,z)\frac{\gamma_1^2\phi_1 + \gamma_2^2\phi_2 + 2(\gamma_1-\gamma_2)\phi_1\phi_2 + \alpha_{12}\big(\gamma_2^2\phi_1^2\phi_2 + \gamma_1^2\phi_1\phi_2^2\big)}{(1+\phi_1+\phi_2+\alpha_{12}\phi_1\phi_2)^2} + xF(t,z)$$

where $\alpha_{12} = (\gamma_1-\gamma_2)^2/(\gamma_1+\gamma_2)^2$, which represents the interaction of two solitary waves for any fixed $z \in (\mathbb{C}^{\mathbb{N}})_c$. Hence, by using the inverse Hermite transformation, we have a stochastic 2-soliton solution of (1.1) in the following

$$U(t,x) = 2A^{\diamond 2}(t,x)\diamond\frac{\gamma_1^2\Phi_1 + \gamma_2^2\Phi_2 + 2(\gamma_1-\gamma_2)\Phi_1\diamond\Phi_2 + \alpha_{12}\big(\gamma_2^2\Phi_1^{\diamond 2}\diamond\Phi_2 + \gamma_1^2\Phi_1\diamond\Phi_2^{\diamond 2}\big)}{\big(1+\Phi_1+\Phi_2+\alpha_{12}\Phi_1\diamond\Phi_2\big)^{\diamond 2}} + xF(t).$$

In a similar manner we can get similar results for equations (3.12)–(3.14), respectively.

(i) An exact solution containing 2-solitary stochastic wave of (3.12) is

$$U_1(t,x) = 2A_1^{\diamond 2}(t,x)\diamond\frac{\gamma_1^2\Phi_1^1 + \gamma_2^2\Phi_2^1 + 2(\gamma_1-\gamma_2)\Phi_1^1\diamond\Phi_2^1 + \alpha_{12}\big(\gamma_2^2\big(\Phi_1^1\big)^{\diamond 2}\diamond\Phi_2^1 + \gamma_1^2\Phi_1^1\diamond\big(\Phi_2^1\big)^{\diamond 2}\big)}{\big(1+\Phi_1^1+\Phi_2^1+\alpha_{12}\Phi_1^1\diamond\Phi_2^1\big)^{\diamond 2}} + x$$

where

$$A_1(t) = \exp\{-6K_2B(t)\}$$

and

$$\Phi_i^1(t,x) = \exp^{\diamond}\left\{\gamma_i x e^{-6K_2B_t} - \gamma_i^3K_2\int_0^t e^{-9K_2(2B(s)-s^2)}\delta B(s) + x_i\right\} \qquad i = 1,2,$$

x_1 and x_2 are any constants.

(ii) An exact solution containing 2-solitary stochastic wave of (3.13) is

$$U_2(t,x) = 2A_2^{\diamond 2}(t,x)\diamond\frac{\gamma_1^2\Phi_1^2 + \gamma_2^2\Phi_2^2 + 2(\gamma_1-\gamma_2)\Phi_1^2\diamond\Phi_2^2 + \alpha_{12}\left(\gamma_2^2\left(\Phi_1^2\right)^{\diamond 2}\diamond 2\Phi_2^2 + \gamma_1^2\Phi_1^2\diamond\left(\Phi_2^2\right)^{\diamond 2}\right)}{\big(1+\Phi_1^2+\Phi_2^2+\alpha_{12}\Phi_1^2\diamond\Phi_2^2\big)^{\diamond 2}} + xB(t)$$

where

$$A_2(t) = \exp\{-3k_{12}(B^2(t) - t)\}$$

and

$$\Phi_i^2(t, x) = \exp^{\diamond}\left\{\gamma_i x e^{-3k_{12}(B^2(t)-t)} - \gamma_i^3 k_{12}\int_0^t e^{-9k_{12}(B^2(s)-s)}\delta B(s) + x_i\right\} \qquad i = 1, 2.$$

(iii) An exact solution containing 2-solitary stochastic wave of (3.14) is

$$U_3(t, x) = 2\frac{\gamma_1^2\Phi_1^3 + \gamma_2^2\Phi_2^3 + 2(\gamma_1 - \gamma_2)\Phi_1^3 \diamond \Phi_2^3 + \alpha_{12}\left(\gamma_2^2\left(\Phi_1^3\right)^{\diamond 2} \diamond \Phi_2^3 + \gamma_1^2\Phi_1^3 \diamond \left(\Phi_2^3\right)^{\diamond 2}\right)}{\left(1 + \Phi_1^3 + \Phi_2^3 + \alpha_{12}\Phi_1^3 \diamond \Phi_2^3\right)^{\diamond 2}}$$

where

$$\Phi_i^3(t, x) = \exp\left\{\gamma_i x - \gamma_i^3 K_2\left(B(t) - \frac{1}{2}t^2\right) + x_i\right\} \qquad i = 1, 2.$$

Remark. Since there is a unitary map between the Wiener white noise space and the Poisson white noise space, we can obtain the solution of the Poissonian SPDE simply by applying this map to the solution of the corresponding Gaussian SPDE. A nice, concise account of this connection was given by Benth and Gjerde in [1]. We can see section 4.9 of [8] also. Hence, we can get stochastic single and multi-soliton solutions as we do in section 3 and section 4 if the coefficient $f(t)$ is perturbed by Poissonian white noise in (1.2).

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